Journal of Engineering Mathematics 20: 293–306 (1986)

High- and low-frequency asymptotic consequences of the Kramers-Kronig relations

M. GREENHOW *

Mathematics Department, University of Manchester, Manchester M13 9PL, England

(Received September 14, 1985 and in final form June 6, 1986)

Summary

This paper examines the high- and low-frequency asymptotic consequences of the Kramers-Kronig relations which hold for the real and imaginary parts of the Fourier transform of the response of any linear causal system. By knowing or assuming the high- or low-frequency expansion of one of these functions, the corresponding asymptotic expansion of the other is easily determined, and furthermore the coefficients in this expansion may be determined by evaluating integrals that involve intermediate frequencies. The practical utility of the approach is demonstrated with examples from floating-body hydrodynamics and some new integral relations are derived and exploited.

1. Introduction

For the wide class of linear causal systems it is well known that the real and imaginary parts of the Fourier transform of the response function are not unrelated but satisfy the Kramers-Kronig relations (see Landau and Lifschitz [1], pp. 396–397). With a simple change of variable these relations reduce to one-sided Hilbert transforms, equations (1) and (2) in this paper, whose asymptotic consequences are discussed here for both high and low frequencies. For illustration the paper takes examples from floating-body hydrodynamics, where the added mass and damping are related by the Kramers-Kronig relations, although it is stressed that the method is generally applicable. By assuming the first few terms in the high-frequency expansion of the damping, the form of the high-frequency expansion of the added mass may be derived quite easily from (1) using Mellin transforms (see Ursell [2]). Some of the coefficients in this expansion are known explicitly whilst others may be calculated from integrals over the damping if they converge sufficiently well. The practical utility (or otherwise) of these integrals is demonstrated for the cases chosen. Furthermore, use of (1) and (2) together yields new integral relations for the added mass, which again may be exploited.

Besides demonstrating the general method, the added mass and damping example is strongly motivated because direct numerical computation of these frequency-dependent parameters becomes impractical much beyond dimensionless wavenumber β of 10 even

* Present address: Norwegian Marine Technology Research Institute, P.O. Box 4125, Valentinlyst, N-7001 Trondheim, Norway

for special geometries (e.g. the sphere or cylinder chosen here). The alternative analytical approach of expanding the high-frequency problem gives useful results (see Simon [3]) although the method becomes increasingly difficult to apply for further terms in the expansion. It therefore seems appropriate to extract as much information as possible about the high-frequency behaviour from the Kramers-Kronig relations using the Mellin-transform method described here. The low-frequency asymptotic consequences may also be explored with the same method. However, since numerical or analytical methods (see Simon and Hulme [4]) work well for the low-frequency floating-body problem there seems little advantage in exploiting the new relationships derived.

2. The Kramers-Kronig relations and their direct use

When a body is forced to oscillate in an incompressible, inviscid and homogeneous fluid with a free surface, the fluid exerts a force on the body that is not generally in phase with the body acceleration. It is usual to define a dimensionless added-mass parameter P_m and a dimensionless damping parameter P_d as follows:

$$P_m = \frac{\text{component of force } 180^\circ \text{ out of phase with body acceleration}}{(\text{acceleration amplitude}) \cdot (\text{mass of fluid displaced})},$$

$$P_d = \frac{\text{component of force } 180^\circ \text{ out of phase with the body velocity}}{(\text{acceleration amplitude}) \cdot (\text{mass of fluid displaced})}.$$

Both quantities are dependent upon the dimensionless wavenumber of oscillation $\beta = Ka$ where K is the wavenumber and a is a typical body dimension (here chosen to be the radius of the cylinder or sphere). The wavenumber and frequency of oscillation are related by the dispersion relation (see Newman [5], chapter 6).

Kotik and Mangulis [6] where the first to point out that these quantities are related by the Kramers-Kronig relations:

$$P_m(\beta) - P_m(\infty) = \frac{1}{\pi} ⨍_0^\infty \frac{P_d(t)\, \mathrm{d}t}{t - \beta}, \tag{1}$$

$$P_d(\beta) = \frac{\sqrt{\beta}}{\pi} ⨍_0^\infty \frac{[P_m(\infty) - P_m(t)]}{\sqrt{t}\,(t - \beta)} \mathrm{d}t, \tag{2}$$

where $⨍$ denotes a principal-value integral. For ease of exposition, let us temporarily suppose that the first few terms in the asymptotic expansion of the damping coefficient are of the form (but see equation (32) later)

$$P_d(\beta) \sim \sum_{n=1}^{N} \frac{a_n}{\beta^n} \quad \text{as } \beta \to \infty. \tag{3}$$

By use of Havelock's [7] wavemaker theory it is not difficult to find examples of flexible wavemaker motions that have high-frequency damping expansions of this form. For the half-immersed cylinder or sphere, used as examples here, the first term in the high-frequency expansion may also be calculated by wavemaker theory and the arguments used in Hulme [8], which show that in heave these bodies have a leading term a_4/β^4, while in sway the leading term is a_2/β^2. The next term in these expansions is actually logarithmic (its effect is discussed later) but this only changes the order of the error terms in the following (see later).

Using Mellin transforms, the arguments used by Wong [9] and Ursell [2] then show that (1) and (3) give

$$\pi[P_m(\beta)-P_m(\infty)] \sim -\sum_{n=1}^{N}\frac{\alpha_n}{\beta^n}-\ln\beta\sum_{n=1}^{N}\frac{a_n}{\beta^n}$$

as $\beta\to\infty$, where

$$\alpha_n=\int_0^1 t^{n-1}\left[P_d(t)-\sum_{k=1}^{n-1}\frac{a_k}{t^k}\right]\mathrm{d}t+\int_1^\infty t^{n-1}\left[P_d(t)-\sum_{k=1}^{n}\frac{a_k}{t^k}\right]\mathrm{d}t,\quad n>1,$$

and

$$\alpha_1=\int_0^1 P_d(t)\,\mathrm{d}t+\int_1^\infty\left(P_d(t)-\frac{a_1}{t}\right)\mathrm{d}t,\quad n=1. \tag{4}$$

Kotik and Mangulis [6] assume that $a_1=0$ and therefore

$$\alpha_1=\int_0^\infty P_d(t)\,\mathrm{d}t,$$

as stated in Kotik and Mangulis and exploited by Greenhow [10] to give the high-frequency leading-order behaviour of the added mass. Wong's expression (equation (4)) gives the form of the expansion, and it is remarkable that the coefficients of logarithmic terms in the added mass are known from the (assumed known) coefficients in (3), the high-frequency behaviour of the damping. For example, for a swaying cylinder we have $a_1=0$, $a_2=8/\pi$, giving

$$P_m(\beta)-P_m(\infty)=-\frac{\alpha_1}{\pi\beta}-\frac{8}{\pi^2}\left(\frac{\ln\beta}{\beta^2}\right)-\frac{\alpha_2}{\pi\beta^2}+\dots\quad\text{as }\beta\to\infty$$

with

$$\alpha_1=\int_0^\nu P_d(t)\,\mathrm{d}t+\frac{a_2}{\nu}+\mathrm{O}(\nu^{-2}),\qquad \alpha_2=\int_0^\nu tP_d(t)\,\mathrm{d}t-a_2\ln\nu+\mathrm{O}(\nu^{-1}). \tag{5}$$

Here we assume that we know the damping $P_d(t)$ for $0\leqslant t\leqslant\nu$ with $\nu>1$ and, for the error term, that the series (3) is valid for the first two terms.

For a heaving cylinder $a_1=a_2=a_3=0$ and $a_4=32/\pi$, giving

$$P_m(\beta)-P_m(\infty)=-\frac{\alpha_1}{\pi\beta}-\frac{\alpha_2}{\pi\beta^2}-\frac{\alpha_3}{\pi\beta^3}-\frac{32}{\pi^2}\frac{\ln\beta}{\beta^4}-\frac{\alpha_4}{\pi\beta^4}+\dots\quad\text{as }\beta\to\infty$$

with

$$\alpha_1 = \int_0^\nu P_d(t)\,\mathrm{d}t + \frac{a_4}{3\nu^3} + \mathrm{O}(\nu^{-4}), \qquad \alpha_2 = \int_0^\nu tP_d(t)\,\mathrm{d}t + \frac{a_4}{2\nu^2} + \mathrm{O}(\nu^{-3}),$$

$$\alpha_3 = \int_0^\nu t^2P_d(t)\,\mathrm{d}t + \frac{a_4}{\nu} + \mathrm{O}(\nu^{-2}), \qquad \alpha_4 = \int_0^\nu t^3P_d(t)\,\mathrm{d}t - a_4 \ln \nu + \mathrm{O}(\nu^{-1}), \quad (6)$$

again with the same assumptions for the error-term estimation.

3. Use of equation (2)

Let us consider use of (2) with the asymptotic form of (5). Then we are required to examine

$$f(x) = \int_0^\infty \frac{g(t)}{t-x}\,\mathrm{d}t \qquad (7)$$

with

$$g(t) = \pi[P_m(\infty) - P_m(t)]/\sqrt{t} \sim \frac{\alpha_1}{t^{1+1/2}} + a_2\frac{\ln t}{t^{2+1/2}} + \frac{\alpha_2}{t^{2+1/2}} \quad \text{as } t \to \infty, \qquad (8)$$

or generally

$$g(t) \sim \sum_{k=1}^{N} \alpha_k/t^{k+1/2} + \sum_{k=1}^{N} a_k \ln t/t^{k+1/2} \quad \text{as } t \to \infty.$$

Taking Mellin transforms, Ursell [11] shows that

$$F(s) = G(s)\pi \cot \pi s \qquad (9)$$

with

$$F(s) = \int_0^\infty x^{s-1} f(x)\,\mathrm{d}x.$$

The inverse transform gives

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-s} G(s)\pi \cot \pi s\,\mathrm{d}s \qquad (10)$$

where c is any constant chosen such that the contour lies to the left of any singularities of the integrand. In this case $c < 1$, and we choose $\frac{1}{2} < c < 1$ for definiteness when moving the integration contours later.

To investigate the high-frequency behaviour of $f(x)$, we move the path of integration to the right across the poles determined by

$$G(s) = \int_0^\infty g(t)t^{s-1}\,\mathrm{d}t$$

$$= \int_0^1 g(t)t^{s-1}\,\mathrm{d}t + \int_1^\infty \left[g(t) - \frac{\alpha_1}{t^{3/2}} - \frac{a_2 \ln t}{t^{5/2}} - \frac{\alpha_2}{t^{5/2}}\right] t^{s-1}\,\mathrm{d}t$$

$$- \frac{\alpha_1}{\left(s-1-\frac{1}{2}\right)} - \frac{\alpha_2}{\left(s-2-\frac{1}{2}\right)} + \frac{a_2}{\left(s-2-\frac{1}{2}\right)^2}. \tag{11}$$

(Term-by-term integration is valid for $\mathrm{Re}(s) < 3/2$. Equation (11) is the analytic continuation of $G(s)$ for $\mathrm{Re}(s) < 7/2$.)

$$\pi \cot \pi s = \pi \frac{\cos[\pi(s-n)]}{\sin[\pi(s-n)]} = (s-n)^{-1} + \mathrm{O}(s-n) \tag{12}$$

and

$$x^{-s} = \frac{1}{x^\alpha} x^{-(s-\alpha)} = \frac{1}{x^\alpha}[1 - (s-\alpha)\ln x + \ldots)]. \tag{13}$$

Substituting (11)–(13) into (10) and moving the contour of integration to the right to extract the high-frequency behaviour give

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c+2-\mathrm{i}\infty}^{c+2+\mathrm{i}\infty} x^{-s} G(s) \pi \cot \pi s\,\mathrm{d}s - (\text{res at } 1) - (\text{res at } 3/2)$$

$$- (\text{res at } 2) - (\text{res at } 5/2)$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{c+2-\mathrm{i}\infty}^{c+2+\mathrm{i}\infty} x^{-s} G(s) \pi \cot \pi s\,\mathrm{d}s - \frac{G(1)}{x} - \frac{G(2)}{x^2}$$

$$+ \pi \cot \frac{\pi}{2} \left[\frac{\alpha_1}{x^{3/2}} + \frac{\alpha_2}{x^{5/2}}\right] + \frac{a_2 \pi^2}{x^{5/2}},$$

the last term arising from the double pole at $s = 5/2$. The general result is as follows: if

$$g(t) \sim \sum_{n=1}^{N} \frac{\alpha_n}{t^{n+1/2}} + \sum_{n=1}^{N} \frac{a_n \ln t}{t^{n+1/2}} \quad \text{as } t \to \infty,$$

then

$$f(x) = \int_0^\infty \frac{g(t)}{t-x}\,\mathrm{d}t \sim -\sum_{n=1}^{N} \frac{G(n)}{x^n} + \pi^2 \sum_{n=1}^{N} \frac{a_n}{x^{n+1/2}} \quad \text{as } x \to \infty. \tag{14}$$

Applying (14) to (2) we obtain

$$P_d(\beta) \sim \sum_{n=1}^{N} \left[\frac{\alpha_n}{\beta^n} - \frac{G(n)}{\pi^2 \beta^{n-1/2}} \right] \quad \text{as } \beta \to \infty.$$

Thus equation (3) for the damping is recovered provided that

$$G(n) \equiv 0 \text{ for all } n \text{ such that } N \geqslant n \geqslant 1. \tag{15}$$

Here $G(s)$ is to be identified with its analytic continuation for $s \geqslant 3/2$, i.e.

$$G(s) = \int_0^1 g(t) t^{s-1}\,\mathrm{d}t + \int_1^\infty \left\{ g(t) - \sum_{k=1}^{m-1} \frac{\alpha_k}{t^{k+1/2}} - \sum_{k=1}^{m-1} \frac{a_k \ln t}{t^{k+1/2}} \right\} t^{s-1}\,\mathrm{d}t$$
$$- \sum_{k=1}^{m-1} \frac{\alpha_k}{\left(s - k - \frac{1}{2}\right)} + \sum_{k=1}^{m-1} \frac{a_k}{\left(s - k - \frac{1}{2}\right)^2}.$$

Here $m > s - \frac{1}{2}$ and integer, thus ensuring that the integral term converges.

Of course, the validity of (15) relies on the validity of the expansion in (3). However, considering the case of a swaying sphere or cylinder, we know to leading order $P_d(\beta) \sim \mathrm{O}(\beta^2)$ as $\beta \to \infty$. Thus

$$G(1) = \int_0^\infty \frac{P_m(\infty) - P_m(t)}{\sqrt{t}}\,\mathrm{d}t = 0 \tag{16}$$

and

$$G(2) = \pi \int_0^1 [P_m(\infty) - P_m(t)] \sqrt{t}\,\mathrm{d}t + \pi \int_1^\infty \left[P_m(\infty) - P_m(t) - \frac{\alpha_1}{\pi t} \right] \sqrt{t}\,\mathrm{d}t - 2\alpha_1$$
$$= 0. \tag{17}$$

Equation (16) was first pointed out by Kotik and Lurye [12] and utilised by Greenhow [10] whilst (17) appears to be new information. For heaving cylinders or spheres $P_d(\beta) \sim \mathrm{O}(\beta^{-4})$ as $\beta \to \infty$. Consequently (15) is certainly true when $N = 1$, 2, 3 and 4 in this case. Explicitly we have (16) and (17) and in addition:

$$G(3) = \pi \int_0^1 [P_m(\infty) - P_m(t)] t^{3/2}\,\mathrm{d}t + \pi \int_1^\infty \left[P_m(\infty) - P_m(t) - \frac{\alpha_1}{\pi t} - \frac{\alpha_2}{\pi t^2} \right] t^{3/2}\,\mathrm{d}t$$
$$- \frac{2\alpha_1}{3} - 2\alpha_2 = 0, \tag{18}$$

$$G(4) = \pi \int_0^1 [P_m(\infty) - P_m(t)] t^{5/2}\,\mathrm{d}t$$
$$+ \pi \int_1^\infty \left[P_m(\infty) - P_m(t) - \frac{\alpha_1}{\pi t} - \frac{\alpha_2}{\pi t^2} - \frac{\alpha_3}{\pi t^3} \right] t^{5/2}\,\mathrm{d}t - \frac{2\alpha_1}{5} - \frac{2\alpha_2}{3} - 2\alpha_3$$
$$= 0. \tag{19}$$

4. Practical utility of equations (16)–(19) and numerical results

The practical utility of (16) has already been demonstrated by Greenhow [10] who manipulates the equation to give

$$2P_m(\infty) = P_m(\nu) + \frac{1}{2\sqrt{\nu}} \int_0^\nu \frac{P_m(t)}{\sqrt{t}} \, dt + O\left(\frac{\ln \nu}{\nu^2}\right) + O\left(\frac{1}{\nu^2}\right). \tag{20}$$

Here it is assumed that $P_m(t)$ is known for $0 \leqslant t \leqslant \nu$ and unknown for $t > \nu$ but that (8) holds for $t > \nu$ and $a_1 = 0$. The logarithmic error term is absent if $a_2 = 0$. It is remarkable that this equation gives the values of $P_m(\infty)$ quite accurately for cases of heaving or swaying spheres even for a limited range of known values of $P_m(t)$. For example, for a swaying sphere $P_m(\nu)$ with $\nu = 5.0$ is only about half of its infinite-frequency value, but using relation (20) gives $P_m(\infty)$ to within 4% error. It is also noted in Greenhow [10] that further terms in the high-frequency expansion of the added mass cannot be obtained reliably from integrals of the added mass over a finite frequency range using (16). Making use of the asymptotic high-frequency expansion of the added mass (equation (8)), and after some manipulation, (17) can be written as

$$\alpha_1 = \frac{2\nu}{3}\pi P_m(\infty) - \frac{\nu\pi P_m(\nu)}{2} - \frac{\pi}{4\sqrt{\nu}} \int_0^\nu P_m(t)\sqrt{t} \, dt + \frac{a_2}{\nu} + O(\ln \nu/\nu^2) + O(\nu^{-2}). \tag{21}$$

Although (21) contains error terms of the same order as Greenhow's [10] equation (14), the numerical results from (21) converge faster since they contain new information about the high-frequency behaviour of the added mass. We note that if the value of $P_m(\infty)$ calculated from (20) is used in (21), then the leading-error term will be O(ln ν/ν) or O($1/\nu$), but sometimes $P_m(\infty)$ is known explicitly from related problems with a simpler free-surface condition (see, e.g., Newman [5], pp. 297–298). Notice also that a_2, the leading coefficient in the damping at high frequency, is involved in (21), but this is assumed to be known. If, however, $P_m(\infty)$ calculated from (20) is used in (21) then it is not consistent to include the term in a_2.

Equation (18) yields the next coefficient α_2 as

$$\alpha_2 = \frac{3P_m(\infty)\pi\nu^2}{5} - \frac{\pi}{4\sqrt{\nu}} \int_0^\nu P_m(t) t^{3/2} \, dt - \frac{2\alpha_1}{3}\nu - \frac{\nu^2 P_m(\nu)}{2}\pi$$
$$+ O(\ln \nu/\nu^2) + O(1/\nu^2), \tag{22}$$

whilst (19) gives

$$\alpha_3 = \frac{4\pi}{7} P_m(\infty)\nu^3 - \frac{\nu^3 P_m(\nu)\pi}{2} - \frac{\pi}{4\sqrt{\nu}} \int_0^\nu P_m(t) t^{5/2} \, dt$$
$$- \tfrac{3}{5}\alpha_1\nu^2 - \tfrac{2}{3}\alpha_2\nu + \frac{a_4}{\nu} + O\left(\frac{\ln \nu}{\nu^2}\right) + O\left(\frac{1}{\nu}\right). \tag{23}$$

Again a_4 is assumed to be known.

We note that using previously calculated values of $P_m(\infty)$ from (20) and α_1 from (21), gives leading-order error in (22) as O(ν^{-1}), whilst using α_2 calculated from (22) gives

Table 1. Coefficients in the expansion of the added mass of a half-submerged heaving circular cylinder

ν	$\frac{\alpha_1}{\pi}$ (Eq. (6))	$\frac{\alpha_2}{\pi}$ (Eq. (6))	$\frac{\alpha_3}{\pi}$ (Eq. (6))	$\frac{\alpha_4}{\pi}$ (Eq. (6))	$P_m(\infty)$ (Eq. (20))	$\frac{\alpha_1}{\pi}$ (Eq. (21))	$\frac{\alpha_2}{\pi}$ (Eq. (22))	$\frac{\alpha_3}{\pi}$ (Eq. (23))
	Damping integrals				Added-mass integrals			
1	1.3871	1.7311	3.3050	0.04303	1.0985	0.24496	−0.04886	2.9602
2	0.51369	0.61656	1.8316	−1.9806	1.0230	0.37841	0.11374	1.3765
3	0.44292	0.45023	1.4355	−2.9362	1.0162	0.41682	0.22648	1.0220
4	0.42976	0.40568	1.2838	−3.4568	1.0161	0.42781	0.29018	0.94271
5	0.42624	0.39020	1.2154	−3.7606	1.0163	0.43069	0.32635	0.94287
6	0.42508	0.38392	1.1812	−3.9465	1.0162	0.43105	0.34778	0.96735
7	0.42465	0.38113	1.1633	−4.0618	1.0159	0.43036	0.35921	0.98534
8	0.42447	0.37978	1.1532	−4.1370	1.0155	0.42979	0.36804	1.0169
9	0.42439	0.37911	1.1476	−4.1844	1.0151	0.42900	0.37205	1.0297
10	0.42435	0.37876	1.1443	−4.2157	1.0147	0.42827	0.37398	1.0336
Shanks ∞	0.4243	0.378	1.14	−4.3	–	–	0.376	1.04
Correction applied ∞	–	–	–	−3.7	–	–	–	–
Simon ∞	0.42441	0.37886	1.14971	−3.8210	1	0.42441	0.37886	1.14197

Table 2. Coefficients in the expansion of the added mass of a half-submerged swaying circular cylinder

ν	$\frac{\alpha_1}{\pi}$ (Eq. (5))	$\frac{\alpha_2}{\pi}$ (Eq. (5))	$P_m(\infty)$ (Eq. (20))	$\frac{\alpha_1}{\pi}$ (Eq. (21))
	Damping integrals		Added-mass integrals	
1	1.0077	0.12252	0.69608	0.75380
2	0.77420	−0.19113	0.48798	0.60940
3	0.77333	−0.28844	0.43615	0.62495
4	0.72409	−0.31951	0.41771	0.65124
5	0.72275	−0.32524	0.41016	0.67224
6	0.72361	−0.32044	0.40683	0.68750
7	0.72502	−0.31127	0.40529	0.69871
8	0.72645	−0.30049	0.40459	0.70697
9	0.72777	−0.28935	0.40430	0.71314
10	0.72891	−0.27845	0.40424	0.71762
Shanks ∞	0.74	–	–	0.73
Correction applied ∞	–	−0.013	–	–
Simon ∞	0.73789	–	0.40529	0.73789

leading-order error in (23) as $O(\ln \nu/\nu)$. Clearly, calculating the coefficient α_n from previously calculated α_i's, $i = 1, \dots, n-1$, will eventually fail to converge; in fact, for the heave case discussed above we cannot generate α_4 by using this method since the leading error is $O(\ln \nu)$. However, in this case it is not established that $G(5) = 0$ and further information about the high-frequency form of the damping would be needed (see later). In the numerical results presented here the exact values of α_i, $i = 1, \dots, n-1$, have been used to calculate α_n, but values obtained from either (5) or (6) do not give serious differences.

It is stressed that in equations (20) to (23) only the first term in the series of (3) is assumed known, whereas actually the next term is logarithmic for the examples chosen (see later). This gives the leading error term as $O(\ln^2\nu/\nu^2)$ in (21) and (23), while in (5) and (6) the error of $O(\nu^{-n})$ is replaced by an error of $O(\ln \nu/\nu^n)$.

Numerical results for a heaving or swaying sphere or cylinder are presented in Tables 1 to 4, showing convergence or otherwise of the formula used. Also shown, where appropriate, are the results of improving the convergence using Shanks transforms (see Bender and Orszag [13], pp. 369–375) and the analytic results of Simon [3] who gives as $\beta \to \infty$:

$$P_d(\beta) \sim \frac{32}{\pi\beta^4}\left[1 + \frac{4}{\pi\beta}(\ln\beta + \gamma + \ln 2 - 3)\right], \tag{24}$$

$$\begin{aligned} P_m(\beta) \sim\ & 1 - \frac{4}{3\pi\beta} - \left(2 - \frac{16}{\pi^2}\right)\frac{1}{\beta^2} - \frac{32}{3\pi^3}\left(4 - \frac{\pi^2}{15}\right)\frac{1}{\beta^3} - \frac{32}{\pi^2}\frac{\ln\beta}{\beta^4} \\ & + \frac{32}{\pi^2\beta^4}\left(\frac{19}{9} + \frac{10}{3\pi^2} - \gamma - \ln 2\right) - \frac{64}{\pi^3}\frac{\ln^2\beta}{\beta^5} \\ & - \frac{128}{\pi^3}(\gamma + \ln 2 - 3)\frac{\ln\beta}{\beta^5} + O\left(\frac{1}{\beta^5}\right) \end{aligned} \tag{25}$$

Table 3. Coefficients in the expansion of the added mass of a half-submerged heaving sphere

ν	$\frac{\alpha_1}{\pi}$ (Eq. (6))	$\frac{\alpha_2}{\pi}$ (Eq. (6))	$\frac{\alpha_3}{\pi}$ (Eq. (6))	$\frac{\alpha_4}{\pi}$ (Eq. (6))	$P_m(\infty)$ (Eq. (20))	$\frac{\alpha_1}{\pi}$ (Eq. (21))	$\frac{\alpha_2}{\pi}$ (Eq. (22))	$\frac{\alpha_3}{\pi}$ (Eq. (23))
	Damping integrals				Added-mass integrals			
1	1.5219	2.1960	4.3283	0.02282	0.5724	0.02449	−0.09064	4.0218
2	0.32131	0.65988	2.2914	−2.7839	0.5056	0.12642	0.00762	1.8033
3	0.21740	0.41517	1.7076	−4.1956	0.4964	0.16840	0.11047	1.2133
4	0.19666	0.34487	1.4677	−5.0198	0.4962	0.18411	0.17973	1.0345
5	0.19072	0.31870	1.3518	−5.5352	0.4973	0.18979	0.22260	0.98952
6	0.18865	0.30746	1.2907	−5.8683	0.4983	0.19164	0.24903	0.99047
7	0.18782	0.30211	1.2562	−6.0909	0.4991	0.19197	0.26527	1.0040
8	0.18746	0.29943	1.2363	−6.2394	0.4996	0.19185	0.27648	1.0273
9	0.18730	0.29807	1.2247	−6.3371	0.4999	0.19151	0.28388	1.0497
10	0.18722	0.29730	1.2175	−6.4058	0.5002	0.19112	0.28897	1.0705
Shanks ∞	–	–	1.2	−6.5	–	–	0.30	–
Correction applied ∞	–	–	–	−5.9	–	–	–	–
Simon ∞	0.1875	0.2983	–	–	0.5	0.1875	0.2983	–

Table 4. Coefficients in the expansion of the added mass of a half-submerged swaying sphere

ν	$\frac{\alpha_1}{\pi}$ (Eq. (5))	$\frac{\alpha_2}{\pi}$ (Eq. (5))	$P_m(\infty)$ (Eq. (20))	$\frac{\alpha_1}{\pi}$ (Eq. (21))
	Damping integrals		Added-mass integrals	
1	0.99642	0.03133	0.5775	0.74729
2	0.64144	−0.44775	0.3878	0.43895
3	0.57119	−0.61584	0.3192	0.43628
4	0.55022	−0.68737	0.2940	0.45730
5	0.54302	−0.71914	0.2831	0.47661
6	0.54052	−0.73264	0.2779	0.49169
7	0.53985	−0.73690	0.2753	0.50309
8	0.53995	−0.73614	0.2739	0.51164
9	0.54036	−0.73265	0.2731	0.51826
10	0.54087	−0.72776	0.2727	0.52340
Shanks ∞	–	−0.74	–	0.54
Correction applied ∞	0.5498	−0.49	–	–
Simon ∞	0.56	–	0.2732	0.56

for a heaving cylinder;

$$P_d(\beta) \sim \frac{8}{\pi\beta^2}\left[1 + \frac{4}{\pi\beta}(\ln\beta + \gamma + \ln 2 - 2)\right], \tag{26}$$

$$P_m(\beta) \sim \frac{4}{\pi^2} - \frac{0.73789}{\beta} - \frac{8}{\pi^2}\left(\frac{\ln\beta}{\beta^2}\right) + \mathrm{O}\left(\frac{1}{\beta^2}\right), \tag{27}$$

for a swaying cylinder;

$$P_d(\beta) \sim \frac{27}{2\beta^4}\left[1 + \frac{4}{\pi\beta}(\ln\beta + \gamma - 1 - 2I_3)\right] \tag{28}$$

with $I_3 = 0.79361$,

$$P_m(\beta) \sim \frac{1}{2} - \frac{3}{16\beta} - \frac{0.29831}{\beta^2} \tag{29}$$

for a heaving sphere, and

$$P_d(\beta) \sim \frac{3}{\beta^2}\left[1 + \frac{4}{\pi\beta}(\ln\beta + \gamma - 1 + I_4)\right] \tag{30}$$

with $I_4 = -0.92056$,

$$P_m(\beta) \sim 0.2732 - \frac{0.56}{\beta} - \frac{3}{\pi}\frac{\ln\beta}{\beta^2} + \mathrm{O}\left(\frac{1}{\beta^2}\right) \tag{31}$$

for a swaying sphere.

Looking at Tables 1–4 we see that the formulas used give good estimates for the coefficients in the expansion of the added mass except for that of the inverse-power term occurring with the logarithmic term, i.e. for α_2/π for sway and α_4/π for heave. Here convergence is slow and its improvement requires either numerical results for the damping byond $\beta = 10$ or further terms in the high-frequency expansion of the damping as shown above. Applying the latter improvement and using Shanks transforms gives the row labelled "correction applied" in Tables 1–4. Convergence is still very slow and little confidence can be placed in these estimates.

5. Further extension at high-frequency and low-frequency asymptotics

From Simon's results we see that beyond the leading order in the high-frequency asymptotics of the damping, there are logarithmic terms. We therefore suppose that

$$P_d(\beta) \sim \sum_{n=1}^{N} \frac{a_n}{\beta^n} + \sum_{n=1}^{N} \frac{b_n \ln \beta}{\beta^n} \quad \text{as } \beta \to \infty. \tag{32}$$

Proceeding as in Section 2 with the use of Mellin transforms, equation (1) implies that

$$\pi[P_m(\beta) - P_m(\infty)] \sim -\sum_{n=1}^{N} \frac{\alpha_n}{\beta^n} - \ln \beta \sum_{n=1}^{N} \frac{a_n}{\beta^n} - \frac{\ln^2\beta}{2} \sum_{n=1}^{N} \frac{b_n}{\beta^n}$$

with

$$\alpha_n = \int_0^1 \left[P_d(t) - \sum_{k=1}^{n-1} \frac{a_k}{t^k} - \sum_{k=1}^{n-1} \frac{b_k \ln t}{t^k} \right] t^{n-1}\, dt$$
$$+ \int_1^\infty \left[P_d(t) - \sum_{k=1}^{n} \frac{a_k}{t^k} - \sum_{k=1}^{n} \frac{b_k \ln t}{t^k} \right] t^{n-1}\, dt - \frac{\pi^2}{3} b_n. \tag{33}$$

It is seen that the logarithmic terms in the damping give rise to $\ln^2\beta/\beta^2$ terms in the added mass, of known coefficient. This is seen to be consistent with (24) and (25).

It would be possible to use (2) with this new form of the added mass at high frequency to give further integral relationships for the added mass as was done in Section 3. In view of the poor convergence of equations (21) for sway and (23) for heave, such new relations are not likely to have any practical utility.

Let us now consider low frequency, where we suppose that

$$P_d(\beta) \sim \sum_{n=0}^{N} a_n \beta^n \quad \text{as } \beta \to \infty. \tag{34}$$

Then applying the technique of Section 2 to (1) (but moving the contour of integration left across the poles) gives

$$\pi[P_m(\beta) - P_m(\infty)] = -\sum_{n=0}^{N} a_n \beta^n \ln \beta + \sum_{n=0}^{N} \alpha_n \beta^n$$

with

$$\alpha_n = \int_0^1 \left[P_d(t) - \sum_{k=1}^{n} a_k t^k \right] t^{-n-1}\, \mathrm{d}t + \int_1^{\infty} \left[P_d(t) - \sum_{k=1}^{n-1} a_k t^k \right] t^{-n-1}\, \mathrm{d}t. \qquad (35)$$

The leading-order term in the added-mass expansion was considered by Kotik and Mangulis [6] in the cases when a_0 is either finite or zero. If $a_0 = 0$ then

$$P_m(0) - P_m(\infty) = \frac{1}{\pi} \int_0^{\infty} \frac{P_d(t)}{t}\, \mathrm{d}t \qquad (36)$$

which is consistent with (35) and was utilised by Greenhow [10].

In principle we could use (35) to calculate the coefficients in the added-mass expansion at low frequencies from knowledge of the damping over the frequency range, or in the range ν to ∞ with ν decreasing until satisfactory convergence is obtained. However, in contrast to the high-frequency case, the low-frequency problem can be solved much more easily (see Simon and Hulme [4]), and in any case numerical methods are known to work well at low frequency. Thus in the practical sence, (35) yields only the form of the expansion and the coefficients of the logarithmic terms, but not the coefficients of the power terms. It is also possible to follow Section 3 and derive relationships for integrals involving the added mass; such results again have little practical significance. Finally it is noted that for some surface-piercing heaving bodies the leading-order term in the damping is very easy to calculate, being simple related to the waterplane area via the hydrostatic force (see Newman [5], p. 303–304, for an explanation and Kotik and Mangulis [6] for the results in the non-dimensional form used here).

6. Conclusion

It has been shown that the knowledge of the high-frequency form of the damping predicts the form of the high-frequency added mass. Some of the coefficients in this series (those for logarithmic terms) are simply related to those in the damping expansion, whilst the others (those for the inverse power terms) require integrals over the damping or added mass. Convergence of such integrals is good (except for the coefficient of the power term occurring with the first logarithmic term) and provides a practical method of extrapolating numerical results to high frequency. Use of both parts of the Kramers-Kronig relations yields new integral relations for the added mass.

Similar remarks also apply to low frequency, but for this hydrodynamic problem low frequency presents no special problems and can be treated directly. The method is, however, directly applicable and may be useful in a wide variety of linear, causal problems, not just to this example in hydrodynamics.

Acknowledgement

I would like to thank Professor F. Ursell and Dr. M. Simon for useful discussions.

7. References

[1] L.D. Landau and E.M. Lifschitz, *Statistical physics*, Addison Wesley, Reading, Mass. (1958).
[2] F. Ursell, Integrals with a large parameter: Hilbert transforms, *Math. Proc. Camb. Phil. Soc.* 93 (1983) 141–149.
[3] M.J. Simon, The high frequency radiation of water waves by oscillating bodies, *Proc. Roy. Soc.* A 401 (1985) 89–115 and private communication.
[4] M.J. Simon and A. Hulme, The radiation and scattering of long water waves, In: *Hydrodynamics of Ocean Wave-Energy Utilisation*, pp. 419–431, Springer, Berlin (1985).
[5] J.N. Newman, *Marine hydrodynamics*, M.I.T. Press, Cambridge, Mass. (1977).
[6] J. Kotik and V. Mangulis, On the Kramers-Kronig relations for ship motions, *Internat. Shipbuilding Prog.* 9 (1962) 361–368.
[7] T.H. Havelock, Forced surface waves on water, *Phil. Mag.* 8 (1929) 569–576.
[8] A. Hulme, The wave forces acting on a floating hemisphere undergoing forced periodic oscillations, *J. Fluid Mech.* 121 (1982) 443–463.
[9] R. Wong, Asymptotic expansion of the Hilbert transform, *S.I.A.M. J. Math. Anal.* 11 (1980) 92–99.
[10] M. Greenhow, A note on the high frequency limits of a floating body, *J. Ship Res.* 28 (1984) 226–228.
[11] F. Ursell, private communication (1984).
[12] J. Kotik and J. Lurye, Some topics in the theory of coupled ship motions, *Proc. 5th Symp. Naval. Hydrodynamics*, Bergen (1964), pp. 407–424.
[13] C. Bender and S. Orszag, *Advanced mathematical methods for scientists and engineers*, McGraw-Hill, New York (1978).