FREQUENCY-DEPENDENT ADDED MASS IN MODELS FOR CONTROLLER DESIGN FOR WAVE MOTION DAMPING

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Abstract: The paper presents a method for generating a time domain formulation of the equations of motion for a ship with frequency-dependent hydrodynamic coefficients. Previous work on this topic has relied on the use of convolution terms, whereas in this work state-space models are used. This leads to a model formulation that is well suited for controller design and simulation. Copyright© 2003 IFAC

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1. INTRODUCTION

The concept of frequency-dependent added mass and potential damping is well established in the formulation of the equations of motion for a surface ship (Ogilvie, 1964). This formulation is used in identification experiments for ship models using motion at a single frequency, and it is used in commercially available programs like WAMIT and VERES. It was shown by (Cummins, 1962) that the frequency dependence of added mass and potential damping can be seen as a consequence of a convolution term in the radiation potential. The convolution term in the radiation potential leads to a convolution term in the equation of motion (Cummins, 1962), (Ogilvie, 1964). For motion at a single frequency the convolution term in the equation of motion is equivalent to frequency-dependent added mass and potential damping parameters. The formulations of the equation of motion based on the use of convolution terms, or, alternatively, on frequency-dependent parameters are not in agreement with the model formulations used in simulation and in automatic control. As a result of this it is not straightforward to apply the usual methods for simulation and for controller analysis and design. This provides the motivation for investigating this problem further.

The main contribution of the present paper is to show how the equations of motion for a surface ship can be reformulated into a state-space form suited for simulation and controller design. Moreover, it is shown that analysis based on the Laplace transformation, state-space models and energy considerations provides additional insight into the radiation problem. We propose a new method that generates a low order state-space model from frequency-dependent added mass and frequency-dependent potential damping as obtained from identification experiments or numerical computations. The resulting model gives an accurate and computationally efficient representation of the convolution term.

2. PRELIMINARIES

2.1 Equation of motion

The usual method for finding the hydrodynamic forces acting on a ship moving in waves is based on the superposition of the radiation forces due to the ship motion in an undisturbed sea, and of the diffraction forces due to the wave forces on a nonmoving ship (Newman, 1977). The velocity potential \( \phi \) is then described as the sum \( \phi = \phi_R + \phi_D \) where the radiation
potential $\phi_R$ is due to the motion of the ship on an undisturbed sea, and the diffraction potential $\phi_D$ is due to the wave excitation. The pressure on the hull is assumed to be given by

$$p = \rho g z - \rho \frac{\partial \phi_R}{\partial t} - \rho \frac{\partial \phi_D}{\partial t}$$ (1)

which is a sum of hydrostatic pressure, radiation pressure and diffraction pressure. The force $F$ and the moment $M$ on the ship hull can then be described as sums of hydrostatic terms, radiation effects and diffraction effects. The radiation force and the radiation moment are given by

$$F^R = -\rho \int_{S_R} n \frac{\partial \phi_R}{\partial t} dA$$ (2)

$$M^R = -\rho \int_{S_R} (r \times n) \frac{\partial \phi_R}{\partial t} dA$$ (3)

For notational simplicity the motion is described with generalized coordinates $q = (q_1, \ldots, q_6)^T$ and associated generalized forces $\tau = (\tau_1, \ldots, \tau_6)^T$. The results can be extended to the usual kinematics as given in (Fossen, 1994). The equation of motion is given by

$$\sum_{k=1}^{6} m_{jk} \ddot{q}_k = \tau_{R}^H + \tau_{R}^D + \tau_{j}^A + \tau_{j}^E$$ (4)

where $m_{jk}$ are the inertia parameters of the ship, $\tau_{R}^H$ are the hydrostatic forces, $\tau_{R}^D$ are the radiation forces, $\tau_{j}^A$ are the diffraction forces, $\tau_{j}^E$ are the actuator forces, and $\tau_{j}^F$ are the external forces.

2.2 Radiation problem with memory effects

Memory effects in the radiation potential are included in the form of convolution terms in the radiation potential (Cummins, 1962). The radiation potential is then

$$\phi_R(r, t) = \sum_{i=1}^{6} \dot{q}_i(t) \psi_i(r) + \sum_{i=1}^{6} \int_{-\infty}^{t} \chi_i(r, t - \tau) \dot{q}_i(\tau) d\tau$$ (5)

The physical interpretation of the potentials $\psi_i(r)$ and $\chi_i(r, t)$ is made clear by considering an impulse in the velocity $\dot{q}_i$, which is equivalent to a step in $q_i$. The resulting radiation potential is

$$\dot{q}_i = \delta(t) \Rightarrow \phi_R(r, t) = \sum_{i=1}^{6} (\delta(t) \psi_i(r) + \chi_i(r, t))$$ (6)

It is seen that the potentials $\psi_i(r)$ represent the instantaneous response of the fluid due to the ship motion, whereas $\chi_i(r, t)$ are impulse responses to the ship velocity. Boundary conditions and further details are found in (Ogilvie, 1964).

2.3 Equation of motion including memory effects

When the radiation potential is given by (5), the radiation force is found to be (Ogilvie, 1964)

$$\tau_{R}^j = \tau_{R}^0 - \sum_{k=1}^{6} a_{jk} \ddot{q}_k - \sum_{k=1}^{6} b_{jk} \dot{q}_k - \sum_{k=1}^{6} c_{jk} q_k - \sum_{k=1}^{6} \int_{-\infty}^{t} K_{jk}(t - \sigma) \ddot{q}_k(\sigma) d\sigma$$ (7)

where $a_{jk}$ are the added mass parameters, $b_{jk}$ are the potential damping parameters, $c_{jk}$ are restoration coefficients, and the convolution terms in the expression for $\tau_{R}^j$ are due to the convolution terms in the radiation potentials (5) (Ogilvie, 1964). It is noted that in (Newman, 1977) the radiation potential is assumed to be $\phi_R = \sum_{i=1}^{6} \dot{q}_i \psi_i$, in which case $K_{jk} = 0$, and the convolution terms can be set to zero.

Suppose that the velocity is a unit impulse $\dot{q}_k(t) = \delta(t)$. Then the convolution terms are given by

$$\int_{-\infty}^{t} K_{jk}(t - \sigma) \delta(\sigma) d\sigma = K_{jk}(t)$$ (8)

This means that $K_{jk}(t)$ is the impulse response function in direction $j$ to an impulse in velocity in direction $k$. Note that if the positions were considered as the inputs, then $K_{jk}(t)$ would be step response functions.

The equation of motion is found to be

$$\sum_{k=1}^{6} (m_{jk} + a_{jk}) \ddot{q}_k + \sum_{k=1}^{6} b_{jk} \dot{q}_k + \sum_{k=1}^{6} c_{jk} q_k + \sum_{k=1}^{6} \int_{-\infty}^{t} K_{jk}(t - \sigma) \ddot{q}_k(\sigma) d\sigma = \tau_{D}^j + \tau_{A}^j$$ (9)

It is noted that the effect of wave excitation is captured by the diffraction force $\tau_{D}^j$, and that wave excitation is not involved in the derivation of the present convolution term. Moreover, it is noted that the derivation of (9) was done without resorting to frequency analysis results. In particular, the concept of frequency-dependent added mass and potential damping has not been introduced so far.

2.4 Single frequency motion

The parameters of the equations of motion can be found from model experiments with single frequency motion $q_k(t) = q_k \cos(\omega t)$ with $q_k(t) = 0$ for $t < 0$. Then, according to standard arguments all terms in the equation of motion will eventually be sinusoidal and of the same frequency $\omega$. In particular we note that the convolution integral can be written (Ogilvie, 1964)
Therefore turn our attention back to the formulation to describe transient dynamics (Cummins, 1962). We added mass and potential damping cannot be used where

\[ q(t) = \int_{-\infty}^{t} K_{jk}(t - \sigma) \dot{q}_k(\sigma) d\sigma \] (10)

\[ \dot{q}_k(t) - \frac{1}{\omega} \int_{0}^{\infty} K_{jk}(\sigma) \sin \omega \sigma d\sigma \] (11)

\[ + \dot{q}_k(t) \left( \int_{0}^{\infty} K_{jk}(\sigma) \cos \omega \sigma d\sigma \right) \] (12)

This leads to the equation of motion in the widely used form

\[ \sum_{k=1}^{6} \left( m_{jk} + \alpha_{jk}(\omega) \right) \ddot{q}_k + \sum_{k=1}^{6} \beta_{jk}(\omega) q_k + \sum_{k=1}^{6} c_{jk} q_k = r^W_j + r^A_j \] (13)

where the frequency-dependent added mass \( \alpha_{jk}(\omega) \) and the frequency-dependent potential damping parameters \( \beta_{jk}(\omega) \) are given by

\[ \alpha_{jk}(\omega) = a_{jk} - \frac{1}{\omega} \int_{0}^{\infty} K_{jk}(t) \sin \omega t dt \] (14)

\[ \beta_{jk}(\omega) = b_{jk} + \int_{0}^{\infty} K_{jk}(t) \cos \omega t dt \] (15)

It is seen that this formulation captures the frequency dependence of the added mass and the potential damping parameters that is observed in identification experiments with single-frequency motion. However, the equation of motion in the form (13) is only valid under the assumption that \( q_k(t) = q_k \cos \omega t \), and that \( r^W_j \) and \( r^A_j \) are sinusoidal functions at frequency \( \omega \). The model formulation (13) with frequency dependent added mass and potential damping cannot be used to describe transient dynamics (Cummins, 1962). We therefore turn our attention back to the formulation based on convolution terms, and develop this further.

### 2.5 Convolution terms from experimental data

The following problem is addressed: Find the impulse functions \( K_{jk}(t) \) when the frequency-dependent parameters \( \alpha_{jk}(\omega) \) and \( \beta_{jk}(\omega) \) are given. The Fourier transformation \( \tilde{K}_{jk}(\omega) \) of the impulse response function \( K_{jk}(t) \)

\[ \tilde{K}_{jk}(\omega) = \int_{-\infty}^{\infty} K_{jk}(t) \cos \omega t dt \] (16)

\[ -j \int_{-\infty}^{\infty} K_{jk}(t) \sin \omega t dt \] (17)

where \( j^2 = -1 \). The impulse response \( K_{jk}(t) \) may be assumed to be of finite energy. Therefore the Fourier transform \( \tilde{K}_{jk}(\omega) \) must converge to zero as the frequency \( \omega \) tends to infinity, that is, \( \lim_{\omega \to \infty} \tilde{K}_{jk}(\omega) = 0 \). This implies that \( \alpha_{jk}(\infty) = a_{jk} \) and \( \beta_{jk}(\infty) = b_{jk} \), and it follows that \( K_{jk}(t) \) can be found from either of the inverse transformations

\[ K_{jk}(t) = -\frac{2}{\pi} \int_{0}^{\infty} \omega \left[ \alpha_{jk}(\omega) - \alpha_{jk}(\infty) \right] \sin \omega t d\omega \] (18)

### 3. LAPLACE TRANSFORMATION AND STATE-SPACE MODEL

#### 3.1 Introduction

So far the established model formulation for the radiation problem of a ship has been discussed. In this section we will present the equation of motion using the Laplace transformation and with a state-space model.

#### 3.2 Laplace transformation of convolution form equation of motion

The Laplace transformation of the convolution integral is

\[ \mathcal{L} \left\{ \int_{-\infty}^{t} K_{jk}(t - \sigma) \dot{q}_k(\sigma) d\sigma \right\} = sK_{jk}(s)q_k(s) \] (19)

where \( K_{jk}(s) = \mathcal{L} \{ K_{jk}(t) \}, q_k(s) = \mathcal{L} \{ q_k(t) \} \), and where it is used that \( \mathcal{L} \{ \dot{q}_k(t) \} = s q_k(s) \). Then the Laplace transformation of the equation of motion (9) is found to be

\[ \sum_{k=1}^{6} \left( (m_{jk} + \alpha_{jk}) s^2 + (b_{jk} + K_{jk}) s \right) q_k(s) + c_{jk} q_k(s) = r^D_j(s) + r^A_j(s) \] (20)

#### 3.3 State space

Convolution term \( j \) may be written

\[ \mu_j = \sum_{k=1}^{6} \int_{-\infty}^{t} K_{jk}(t - \sigma) \dot{q}_k(\sigma) d\sigma \] (22)

Define the vector \( \mu \) and the matrix \( K \) by

\[ \mu = (\mu_1, \ldots, \mu_6)^T, \quad K(s) = \{ K_{ij}(s) \} \] (23)

where \( K_{ij}(s) = \mathcal{L} \{ K_{ij}(t) \} \) is the Laplace transform of the impulse response function \( K_{ij}(t) \). Suppose that we are able to find a state space model with state vector \( \xi \), input \( \dot{q} \) and output \( \mu \) so that

\[ \dot{\xi} = A\xi + B\dot{q} \] (24)

\[ \mu = C\xi + D\dot{q} \] (25)

The equation of motion can then be written
\[
\sum_{k=1}^{6} (m_{jk} + a_{jk}) \ddot{q}_k = -\sum_{k=1}^{6} b_{jk} \dot{q}_k
\]
\[
-\sum_{k=1}^{6} c_{jk} \dot{q}_k - \sum_{k=1}^{6} \mu_{jk} + \tau_{j}^{D} + \tau_{j}^{A}
\]
\[
\dot{\xi}_{jk} = A_{jk} \xi_{jk} + B_{jk} \dot{q}_k
\]
\[
\mu_{jk} = C_{jk} \xi_{jk} + D_{jk} \dot{q}_k
\]

4. ENERGY CONSIDERATIONS

4.1 Passivity of the radiation problem

In this section the passivity properties of the radiation forces will be investigated (\cite{17}). This can be done by assuming that there are no waves, that is, with superposition is assumed. The ship motion given by the velocities \(q_k\) induces fluid motion described by the radiation potential \(\phi_{R}\). The energy
\[
V_R = \frac{1}{2} \sum_{k, j=1}^{6} [a_{jk} \dot{q}_k \dot{q}_j + c_{jk} q_k \dot{q}_j] + \int_{0}^{T} \mu_j(t) \dot{q}_j(t) dt 
\]
\[
\geq 0
\]

of the fluid due to the radiation potential \(\phi_{R}\) is thus supplied to the fluid by interaction with the ship. Energy-flow arguments lead to the conclusion that the time derivative of the energy \(\dot{V}_R\) is equal to the power supplied from the ship motion, that is,
\[
\dot{V}_R = \dot{q}^T \tau_R = \sum_{j=1}^{6} \dot{q}_j \tau_{j}^{R}
\]

This implies that the mapping \(\dot{q} \mapsto \tau_{R}\) is passive.

4.2 Positive realness

The radiation force is given by the vessel motion \(q_k\) according to
\[
\tau_{j}^{R}(s) = \left[ A(s) + B + K(s) + \frac{1}{s} C \right] s q(s)
\]

where
\[
A = \{a_{ij}\}, \quad B = \{b_{ij}\}, \quad K(s) = \{K_{ij}\}, \quad C = \{c_{ij}\}
\]

The passivity of the mapping \(\dot{q} \mapsto \tau_{R}\) implies that the transfer function matrix
\[
H_R(s) = A(s) + B + K(s) + \frac{1}{s} C
\]

is positive real.

5. GENERATION OF APPROXIMATE STATE-SPACE MODEL

In this section we present a new method for the calculation of a low order state-space representation of the convolution term. An approximation \(\mu_{jk}\) of the convolution terms
\[
\mu_{jk} = \int_{0}^{t} K_{jk}(t - \sigma) \dot{q}_k(\sigma) d\sigma
\]

will now be developed in the state-space form
\[
\dot{\xi}_{jk} = A_{jk} \xi_{jk} + B_{jk} \ddot{q}_k
\]
\[
\mu_{jk} = C_{jk} \xi_{jk} + D_{jk} \dot{q}_k
\]

where \(u_k = \dot{q}_k\).

Consider the frequency-dependent added mass \(\alpha_{jk}(\omega)\) and damping \(\beta_{jk}(\omega)\) of a ship, which can be found either from theory or experiment. When using experimental results, these are often given in tabular form, and must be interpolated up to the desired frequency resolution. In addition, the asymptotic values must be calculated, due to the difficulties of finding these experimentally. In this case, \(\alpha_{jk}(\omega)\) and \(\beta_{jk}(\omega)\) are taken from experiments for a model of a 200kWDT tanker (length 310 m, displacement 234826 m\(^3\)) done by Sierevogel (Sierevogel, 1997), where the data was taken from a scaled frequency range of \(\omega' = \omega \sqrt{\frac{L}{g}}\) from 1 to 6. The data acquisition was done by reading point values from curves and linearly interpolating these and extrapolating the asymptotic values. The only known asymptotic values in this work were \(\beta_{jk}(\infty) = 0\). Nonetheless, the resulting errors in the curves for \(\alpha_{jk}(\omega)\) and \(\beta_{jk}(\omega)\) from the lack of accurate values are not significant for this work, as the method is valid for any \(\alpha_{jk}(\omega)\) and \(\beta_{jk}(\omega)\) that are physically consistent.

The impulse responses \(K_{jk}(t)\) is calculated from (18) as the accuracy of the potential damping parameters \(\beta(\omega)\) were better than the added mass parameters in the data material that was used. \(K_{jk}(t)\) is calculated for each \(t\) by trapezoidal integration for each \(t\) spaced by the time-step \(h = 0.01\).
\[
K_{jk}(t) = \frac{\Delta \omega}{\pi} \sum_{i=1}^{n-1} 2 \beta_{jk}(i \Delta \omega) \cos(i \Delta \omega \cdot t)
\]
\[
\quad + \frac{\Delta \omega}{\pi} (\beta_{jk}(0) + \beta_{jk}(n) \cos(n \Delta \omega \cdot t))
\]

This will not introduce any error since \(\beta(\omega)\) is linearly interpolated. Here, the upper limit was taken to be \(T = 20\), as the \(K_{jk}(t)\) converges at approximately \(t = 15\), and using a step-size \(\Delta \omega = 0.01\) to ensure sufficient smoothness.

Using the impulse responses \(K_{jk}(t)\) as computed from (40) we apply the system identification scheme based on the Hankel SVD method proposed in (Kung, 1978), which is readily available as the function...
IMP2SS in the Robust Control Toolbox of MATLAB. This function can take the impulse response \( K_{jk}(t) \) as input and outputs the system matrices \( \tilde{A}_{jk}, \tilde{B}_{jk}, \tilde{C}_{jk} \) and \( \tilde{D}_{jk} \) for the approximated state-space model. To make it computationally efficient, only some evenly temporally spaced values of \( K_{jk}(\Delta t) \) are used as inputs, but densely enough to capture the main dynamics of the response. In this case \( \Delta t = 0.25 \). The output is then scaled with the time-step \( \Delta t \), according to

\[
\tilde{A}_{jk} = \tilde{A}_{jk}, \quad \tilde{B}_{jk} = \tilde{B}_{jk}
\]
\[
\tilde{C}_{jk} = \tilde{C}_{jk} \Delta t, \quad \tilde{D}_{jk} = \tilde{D}_{jk} \Delta t \quad (41)
\]

The usage of only the values of \( K_{jk}(\Delta t) \) may introduce a slight error in the phase, but this seems to be negligible according to the performed calculations, and is not investigated further.

5.1 Model reduction

The MATLAB function IMP2SS has a built-in model reduction option, which is dependent on an optional input parameter “tol” which bounds the \( H^\infty \)-norm of the error between the approximate realization and an exact realization. This turned out to be an unsatisfactory solution to control the accuracy and order of the state-space approximation. Thus, no specific tolerance was stated, resulting in the order of the approximate system reaching as high as 80-90, but with very good accuracy. This can be seen in relation with the number of measurements of \( K_{jk}(t) \) being \( T/\Delta t = 80 \). Since there are 36 different impulse response functions to be approximated for the complete equations of motion, it is obvious that having approximations of that order is not desirable. Therefore model reduction was used in the form of the MATLAB function BALMR from the Robust Control Toolbox, which are based on truncation and Schur methods (Safonov and Chi-ang, 1989). This function is based on balanced reduction. The BALMR function inputs the state-space matrices \( \tilde{A}_{jk}, \tilde{B}_{jk}, \tilde{C}_{jk} \) and \( \tilde{D}_{jk} \) together with the desired order of the new reduced system, and outputs the new state-space matrices \( \tilde{A}_{jk}, \tilde{B}_{jk}, \tilde{C}_{jk} \) and \( \tilde{D}_{jk} \). In this work, models of order 3, 4 and 5 were calculated.

5.2 Results

In Figure 1 – Figure 6 the outputs \( \mu^i_{jk}(t) \) of the approximated systems given an unit impulse as input are plotted together with the original impulse response functions \( K_{jk}(t) \). Here \( i \) in \( \mu^i_{jk} \) denotes the order of the approximated state-space system. In Figure 7 the Nyquist diagram for the diagonal element \( \mu^1_{11} \) for \( i = 3, 4, 5 \) are plotted. Note that \( K_{11}^1(s) \) and \( K_{11}^3(s) \) are positive real, while \( K_{11}^5(s) \) is not positive real. Thus it is recommended to use an approximation for this mode of order 4 or higher. Likewise, \( K_{55}^3(s) \) is not positive real.

6. REFERENCES

Fig. 3. Impulse response of surge-pitch

Fig. 4. Impulse response of heave-heave

Fig. 5. Impulse response of heave-pitch

Fig. 6. Impulse response of pitch-pitch

Fig. 7. Nyquist diagram of $K_{11}(s)$ where $i$ denotes the approximation order. $i = 5$ (solid), $i = 4$ (dashed) and $i = 3$ (dotted).

