State-space modelling of a vertical cylinder in heave

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The (first-order) wave forces on structures interacting with ocean waves may be represented by complex transfer functions in the frequency domain or by convolution integrals in the time domain. The integration kernels are causal for radiation forces, but not necessarily so for excitation (scattering) forces. As a convenient approximation, these integrals may be replaced by a finite-order system of differential equations with constant coefficients. This method is applied to a heaving cylinder with a vertical axis, and good approximations are obtained when the order of the differential equation system is three for the radiation force and five for the excitation force. The corresponding state-space model for the heave motion response due to the incident wave elevation has an order of 10, but it is demonstrated that it is possible to reduce the order to five without increasing the inaccuracy in the heave motion too much. Copyright © 1996 Elsevier Science Ltd.

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1 INTRODUCTION

One subject of ocean engineering is the study of the interaction between ocean waves and oscillating systems, such as ships, ocean platforms and wave-energy converters. Traditionally, the corresponding analysis is carried out in the 'frequency domain', which corresponds to waves and oscillations being 'monochromatic' or 'harmonic'. This means that physical quantities vary sinusoidally with time with a given angular frequency \( \omega \). For linear problems, where the superposition principle is applicable, this frequency-domain approach is more general than it appears, since a real sea-wave state may be considered as a superposition of many monochromatic waves having different frequencies and different directions of propagation.

An alternative method, the so-called time-domain approach for the study of hydrodynamic interactions, was considered by Cummins1 and Wehausen.2,3 The connection between the time-domain and frequency-domain approaches is provided by means of the Fourier transform (or Laplace transform for the particular case of causal systems). To frequency-dependent coefficients (or 'transfer functions') in the frequency domain, there correspond convolution operators (or 'impulse response functions') in the time domain. Note that a transition from the time domain to the frequency domain is usually applied only to those linear systems which are time-invariant.

In many cases of practical interest, the interaction problem may, however, be more or less nonlinear, such as when the waves are large or extreme. Even with small or moderate waves, there may be significant nonlinearities associated with a wave power converter, for instance due to its power take-off device. The superposition principle is not applicable for nonlinear situations. Hence, a correct analysis of the nonlinear parts of the problem has to be considered in the time domain rather than in the frequency domain. Examples of such an analysis are simulation studies of moored ships4 and of tension-leg platforms.5 Another example, related to the research on wave-energy converters, is the study of optimum control of an oscillating body.6 and later similar time-domain analyses were carried out.
for an oscillating water column and also for a system with an oscillating water column in an oscillating structure (see, for instance, Oltedal). In these theoretical studies, the hydrodynamic interaction is represented by convolution integrals. Thus, the total dynamic system, including wave and oscillation, is described in the form of an integro-differential equation.

A convenient method in time-domain analysis is the ‘state-space’ method, which is very much in use in control engineering. Schmiechen and, independently, Booth proposed the use of this method in hydrodynamics. Further, Jefferys applied the method in the analysis of wave-energy converters. These researchers showed that the convolution-integral model of the hydrodynamic interaction might be represented approximately by some small number of first-order linear constant-coefficient differential equations, which replace the convolution integral in the time domain. Such a replacement has also been proposed, independently, by Yu. Jefferys pointed out the problem of deriving the coefficients of the differential equations explicitly from a given impulse response for the hydrodynamic interaction. He applied identification techniques to derive a state-space model for the radiation interaction associated with Salter’s ‘duck’ by using time-domain input and output signals measured experimentally. In Jefferys’ later research, he also used the Bode plot technique in the frequency domain to find an approximation for the coefficients of the differential equations, and applied numerical optimization to a modelling example to improve the accuracy of the coefficients. The Bode plot method required, however, some manual work to find good initial guesses.

Jefferys stated ‘there is no obvious method’ to compute the coefficients of the differential equations intended to replace the convolution integral, and that a possible computation directly in the time domain would give better accuracy. A method which fulfils these needs is applied here.

One possible method of this kind is to find the coefficients of the differential equations (and thus arrive at a state-space description without a convolution integral) by approximating the impulse response function by a combination of trigonometric and exponential functions in the time domain. Although this method is convenient for numerical computation, it requires some manual preparation to achieve a good combination of trigonometric and exponential functions.

Another method which requires less manual preparation, which is also more general, and which has been used in some previous examples, is also applied in this paper. It uses a matrix exponential function to approximate the impulse response function.

Although the radiation interaction is causal, there are cases where non-causal impulse response functions are of practical interest in ocean engineering. One such case is the relationship between the incident wave and the resulting excitation force on a ship or another ocean structure (see, for instance, figure 2 of Oltedal). Another case is related to the optimum control of oscillation for maximum converted wave energy.

In this paper, Section 2 deals with mathematical methods to analyze linear systems with a single input and a single output, and non-causal systems are considered in particular. The subject of Section 3 is the analysis of the response of a heaving body interacting with waves, and in particular the approximate representation of an impulse response function by means of a state-space model of finite order. Subsequently, as an example, a heaving vertical cylinder is studied numerically, and state-space models are constructed, in Sections 4, 5 and 6, for the impulse response functions for the radiation force; the excitation force; and the heave motion, due to the heave velocity, the incident wave elevation and the incident wave elevation, respectively.

## 2 LINEAR SYSTEMS AND STATE-SPACE MODELLING

We consider a linear system which is time-invariant and time-continuous and which has a single input and a single output. Let \( u(t) \) and \( y(t) \) denote the input signal and the output signal (or response), respectively. They are related as follows:

\[
y(t) = \int_{-\infty}^{\infty} h(t - \tau) u(\tau) \, d\tau
\]

where \( h(t) \) is the ‘impulse response function’, for which the Fourier transform \( H(\omega) \) is termed the ‘frequency response function’. The Fourier transforms \( U(\omega) \) and \( Y(\omega) \) of the input and output signals, respectively, are related by

\[
Y(\omega) = H(\omega) U(\omega)
\]

according to the convolution theorem.

If the system is causal, this is, if \( h(t) = 0 \) for \( t < 0 \), then the upper integration limit \( \infty \) in the convolution integral [eqn (1)] may be replaced by \( t \). If we assume, moreover, that the input signal is a ‘causal function’, that is, \( u(t) = 0 \) for \( t < 0 \), we may replace the lower integration limit \( -\infty \) by 0. In such a case we may use the Laplace transform rather than the Fourier transform. Then we have

\[
Y(s) = H(s) U(s)
\]

where \( H(s) \), \( U(s) \) and \( Y(s) \) are the Laplace transforms of \( h(t) \), \( u(t) \) and \( y(t) \), respectively. The function \( H(s) \) is called the ‘transfer function’, which with \( s = i\omega \) is the frequency response function. Note that eqn (3) is also applicable if the input signal is not a ‘causal function’, provided the effect of the input signal prior to \( t = 0 \) is reflected in the initial conditions for \( u(t) \) and its
derivatives at \( t = 0 \) when deriving the Laplace transform, \( U(s) \), of the input. The system must, however, still be causal (i.e. \( h(t) = 0 \) for \( t < 0 \)) for the (one-sided) Laplace transform to be applicable. However, as explained below, it may sometimes be possible that a system which is non-causal can be transformed by means of a time shift to a system which is causal or approximately causal.

Let us consider a non-causal system for which it is possible to find a positive constant \( t_c \) such that \( h(t) \) is zero or negligible for \( t < -t_c \). Then we define

\[ h_c(t) = h(t - t_c) \] (4)

as the 'causulated impulse response function' corresponding to \( h(t) \). The relationship between \( y(t) \) and

\[ y_c(t) = \int_{-\infty}^{\infty} h_c(t - \tau)u(\tau) \, d\tau \] (5)

as obtained from eqns (1) and (4) is

\[ y(t) = y_c(t + t_c) \] (6)

Note that we may replace the upper integration limit \( \infty \) in the integral [eqn (5)] by \( t \). For the linear system represented by the causulated impulse response function \( h_c(t) \) we may apply the usual Laplace transform analysis.

In many cases of practical interest, it is possible to describe the linear system in the time domain by means of linear differential equations with constant coefficients. Then it is convenient to use a state-space description which is represented by the state equation and the output equation

\[ \dot{X}(t) = AX(t) + Bu(t) \] (7)

and

\[ y(t) = CX(t) \] (8)

respectively. Here \( X(t) \) is an \( n \)-dimensional column vector, where \( n \) is the order of the system of differential equations. Also, \( n \) is the number of states (or the number of dimensions of the state-space). The dot denotes differentiation with respect to time. The constant matrices \( A, B \) and \( C \) are the \( n \times n \) state matrix, the \( n \times 1 \) input (or 'control') matrix and the \( 1 \times n \) output matrix, respectively.

From the theory of Laplace and Fourier transforms, it is evident that the models of the impulse response function and the transfer function are equally good descriptions of a linear system. An exact description is also possible with a state-space model of finite order, provided that the system genuinely possesses a finite (possible large) number of states. However, approximations are possible in order to reduce the number of states from a large finite value, or from an infinite number of states, which are required to describe a spatially distributed dynamic system, which is typical in hydrodynamics.

The three above-mentioned models (or descriptions) may be transformed from one to another according to different purposes. A transfer-function model is usually used for analyzing a linear system in the frequency domain, and an impulse response function model is widely used in the field of ocean engineering for simulating and analyzing a system in the time domain. It is rather straightforward to transform both ways between these two models. The third model, the state-space description, which is widely used in control engineering, may be a more convenient model for investigating a dynamic system in ocean engineering. This model is useful, in particular, for system simulation, digital control and system optimization.

The transformation from a state-space model to the corresponding transfer-function model is carried out in a straightforward manner by taking the Laplace transforms of eqns (7) and (8) and then eliminating \( X(s) \). The resulting transfer function is

\[ H(s) = C(sI - A)^{-1}B \] (9)

where \( I \) is the identity matrix. Since the matrices \( A, B \) and \( C \) are constant, it is easy to determine the state vector \( X(t) \) by integrating the system of differential equations [eqn (7)]. The result is

\[ X(t) = e^{A(t - t_0)}X(t_0) + \int_{t_0}^{t} e^{A(t - \tau)}Bu(\tau) \, d\tau \] (10)

and then the following output function is obtained from eqn (8)

\[ y(t) = Ce^{A(t - t_0)}X(t_0) + \int_{t_0}^{t} Ce^{A(t - \tau)}Bu(\tau) \, d\tau \] (11)

In the following, we shall, for convenience, take \( t_0 = 0 \) as the initial time for the solution of the state equation.

The effects of the input signal \( u(t) \) prior to \( t = 0 \) are contained in the first term of the right-hand side of eqn (11). If the input signal has been applied for all previous time, eqn (10) suggests that we might use

\[ X(0) = \int_{-\infty}^{0} e^{-\lambda t}Bu(t) \, dt \] (12)

as the initial state vector. However, when we consider in what follows the impulse response function of a linear system, we shall assume that the input function \( u(t) \) is a 'causal function' [i.e. \( u(t) = 0 \) for \( t < 0 \)]. Moreover, we shall consider this input to be the only cause of the output. Thus, when modelling the impulse response function, we may assume that the initial state vector vanishes, that is \( X(0) = 0 \). Comparison of eqn (11) with eqn (1) then shows that the state-space model corresponds to a causal impulse response function, which, for \( t > 0 \), is given by

\[ h(t) = Ce^{A(t - t_0)}B \] (13)
In contrast to the above direct results given by eqns (9) and (13), it is less straightforward to obtain state-space models corresponding to a given transfer function or a given impulse response function. Whether these functions have been obtained theoretically or experimentally, it may be of great interest, as argued previously, to construct state-space models which represent these system functions, at least approximately. The transformation from a transfer-function model to a state-space model is called 'realization' of the system (see, for instance, Friedland\textsuperscript{1}). For a given transfer function, there may be many different realizations of the same order (number of states) or of different orders. Usually, the realization of the lowest order ('the minimum realization') is applied. Methods are available for constructing state-space models from input-output information in the frequency domain as well as in the time domain.\textsuperscript{21}-\textsuperscript{24} Examples of system realization are presented below (in Sections 4, 5 and 6). These are based on known impulse response functions. In two cases, we shall consider a non-causal impulse response function.

If a state-space model corresponding to the causalized impulse response function \( h_c(t) \), defined by eqn (4), has system matrices \( A, B \) and \( C \), the state-space model corresponding to the non-causal impulse response function \( h(t) \) can be expressed by the equations

\[
X_c(t) = AX_c(t) + Bu(t) \quad \text{with} \quad X_c(0) = 0 \quad (14)
\]

\[
y(t) = CX_c(t + t_c) \quad (15)
\]

Notice here the time shift \( t_c \) as compared with eqn (8). This means that the output \( y(t) \) at time \( t \) is influenced by the input at a future time \( t + \Delta t \), where \( \Delta t < t_c \). See also eqns (5) and (11).

### 3 THE APPROXIMATION OF A CONVOLUTION INTEGRAL BY A STATE-SPACE MODEL

Considering the heaving motion of a floating body, we shall assume small amplitude and neglect friction. Hence linear theory may be adopted. Newton's law gives the following equation of motion for the heaving body.

\[
M\ddot{z}(t) = f_x(t) + f_r(t) + f_s(t) \quad (16)
\]

where \( M \) and \( \ddot{z}(t) \) are the mass and heave acceleration of the body, respectively. Further, \( f_x(t), f_r(t) \) and \( f_s(t) \) are the excitation force due to the incident wave, the reaction force due to wave radiation and the restoring force acting on the body, respectively.

Taking the Fourier transform of eqn (16), the equation of motion can, in the frequency domain, be written as

\[
-w^2MZ(\omega) = F_x(\omega) + F_r(\omega) + F_s(\omega) \quad (17)
\]

where \( Z(\omega) \) is the frequency response function of the heave displacement of the body.

In linear theory, the excitation force \( F_x(\omega) \) is proportional to the incident wave elevation \( \eta(x, \omega) = A(\omega)e^{-ikx} \), where \( A(\omega) \) is the real and positive amplitude of the incident wave elevation, \( K \) is the angular wave number (or 'angular repetency'), and \( x \) is the horizontal coordinate in the direction of wave propagation. We assume that the origin is at the vertical axis through the centre of the body. Introducing the complex coefficient of proportionality \( H_f(\omega) \) ('the heave excitation force coefficient'), the excitation force is expressed as

\[
F_x(\omega) = H_f(\omega)\eta(0, \omega) = H_f(\omega)A(\omega) = H_f(\omega)e^{i\theta_f(\omega)}A(\omega) \quad (18)
\]

where \( H_f(\omega) \) and \( \theta_f(\omega) \) are the amplitude and phase of the excitation force coefficient, respectively (We use the notation \( H_f(\omega) \equiv |H_f(\omega)| \). The phase \( \theta_f(\omega) \) is relative to the phase of the undisturbed incident wave elevation at \( x = 0 \). Because the excitation force is independent of the motion of the body, instead of the incident wave, the excitation force may, for convenience, be considered as the input to the body.

The reaction force from wave radiation is due to the oscillating velocity of the body and may be expressed as

\[
F_r(\omega) = -[N(\omega) + iwm(\omega)]Z(\omega) = -Z_R(\omega)Z(\omega) \quad (19)
\]

where \( N(\omega) \) is the 'radiation resistance' (or wave damping coefficient) and \( m(\omega) \) is the added mass of the body. The complex coefficient \( Z_R(\omega) \) may be termed the 'radiation impedance',\textsuperscript{25,26} and its imaginary part \( wnz(\omega) \) may be called 'radiation reactance' (a term which need not be confusing even in cases where the 'added mass' is negative\textsuperscript{27}). Note that \( Z(\omega) = iwZ(\omega) \) is the velocity frequency response function of the body.

The restoring force is proportional to the displacement of the body, and is given by

\[
F_s(\omega) = -\rho gSZ(\omega) \quad (20)
\]

where \( \rho \) is the density of water, \( g \) is the gravitational acceleration, and \( S \) is the water plane area of the floating body.

Using eqns (18)-(20) in eqn (17), the frequency domain equation of the heaving body becomes

\[
\{-w^2[M + m(\omega)] + iwN(\omega) + \rho gS\}Z(\omega) = F_x(\omega) \quad (21)
\]

and the frequency response function \( Z(\omega) \) of the heave displacement can be expressed as

\[
Z(\omega) = Z(\omega)e^{i\theta_f(\omega)} = \frac{F_x(\omega)}{-w^2[M + m(\omega)] + iwN(\omega) + \rho gS} \quad (22)
\]

where \( Z(\omega) \) and \( \theta_f(\omega) \) are the amplitude and phase of the heave displacement, respectively.

Taking the inverse Fourier transform of eqn (21), the equation of motion in the time domain may be
expressed as
\[(M + m_\infty) \ddot{z}(t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \{N(\omega) + i\omega[m(\omega) - m_\infty]\} \]
\[\times \hat{Z}(i\omega)e^{i\omega t} d\omega + \rho gSz(t)\]
\[= (M + m_\infty) \ddot{z}(t) + \int_{-\infty}^{\infty} h_R(t - \tau) \dot{z}(\tau) d\tau + \rho gSz(t)\]
\[= f_c(t) \quad (23)\]

where \(h_R(t)\) is the inverse Fourier transform of the reduced radiation impedance \(H_R(i\omega) = N(\omega) + i\omega[m(\omega) - m_\infty]\), and it is the impulse response function of the radiation. Usually, at infinite frequency \((\omega \to \infty)\), the radiation resistance \(N(\omega)\) tends to zero, whereas the added mass \(m(\omega)\) tends to a finite constant \(m_\infty\), which we shall call the 'frequency-independent added mass'. In order to avoid the divergence of the first integral term in eqn (23), we have taken a term \(m_\infty \ddot{z}(t)\) outside the integral. Sometimes numerical methods produce inaccurate results for the radiation impedance at high frequencies, and correspondingly inaccuracies in the early part of the impulse response function may result. Improved results are obtained for this, and also for \(m_\infty\), if specially adapted high-frequency analyses\(^{28-30}\) are used in addition to the usual methods which work satisfactorily at lower frequencies.

It can be shown that the real and imaginary parts of the radiation impedance are even and odd functions of \(\omega\), respectively, i.e. both \(N(\omega)\) and \(m(\omega)\) are even. Using the properties of Fourier transformation, this follows from the fact that \(h_R(t)\) is a real function. Thus \(h_R(t)\) may be expressed as
\[h_R(t) = h_e(t) + h_o(t) \quad (24)\]
where
\[h_e(t) = \frac{1}{\pi \int_{-\infty}^{\infty} N(\omega) \cos(\omega t) d\omega}\]
and
\[h_o(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \omega[m(\omega) - m_\infty] \sin(\omega t) d\omega \quad (25)\]
are even and odd functions of \(t\), respectively.

As the radiation reaction force is caused by the heaving velocity of the body, the corresponding impulse response function is causal,\(^1\) that is \(h_R(t)\) vanishes for \(t < 0\), whereas
\[h_R(t) = 2h_e(t) = 2h_o(t) \quad (27)\]
for \(t > 0\).

Also, considering the causality of \(h_R(t)\), eqn (23) becomes
\[(M + m_\infty) \ddot{z}(t) + \int_{-\infty}^{\infty} h_R(t - \tau) \dot{z}(\tau) d\tau + \rho gSz(t) = f_c(t) \quad (28)\]

This integro-differential equation\(^2\) is termed the impulse response function model of the heave motion and is widely used to simulate the motion of the body in the time domain (see, for instance Wehausen\(^3\)) and Greenhow and Nichols\(^{31}\). In order to represent a convolution integral [as in eqn (28)] approximately by means of a finite-order differential equation system we propose the following method, which we consider to be better than the methods considered by Jefferys.\(^1,14\) To model the heave motion of the body in a state-space description, let us approximate the integral term in eqn (28) by a linear sub-system as follows. The output \(y_p(t)\) of the sub-system is an approximation to the integral term, and the input \(u_p(t)\) of the sub-system is the heave velocity, \(\dot{z}(t)\). Thus, we set
\[X_p(t) = A_pX_p(t) + B_pu_p(t) \quad (29)\]
and
\[y_p(t) = C_pX_p(t) \approx \int_{-\infty}^{t} h_R(t - \tau) \dot{z}(\tau) d\tau \quad (30)\]
where \(X_p(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T\) is the state vector of the sub-system.

Among the many possible realizations of the state-space model, we shall use he so-called companion-form realization, where the matrices \(A_p, B_p, \) and \(C_p\) are of the form

\[A_p = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_1 \\ 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 1 & \cdots & 0 & -a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_n \end{bmatrix} \quad (31)\]

\[B_p = [b_1 \ b_2 \ b_3 \ \ldots \ b_{n-1} \ b_n]^T \quad (32)\]
and
\[C_p = [0 \ 0 \ 0 \ \ldots \ 0 \ 1] \quad (33)\]

Thus we have to choose an integer \(n\) and then determine \(2n\) unknown parameters such that the approximation in eqn (30) is acceptable. The chosen form of the matrix \(C_p\) means that \(x_n(t) = y(t)\).\(^{24,32}\)

For the state-space model, the impulse response function as given by eqn (13) is to approximate the actual impulse response function \(h_R(t)\). Thus we determine the \(2n\) unknown parameters by minimizing the following target function
\[Q = \sum_{k=1}^{m} G(t_k)[h_R(t_k) - C_p e^{A_p t_k} B_p]^2 \quad (34)\]
where \(G(t_k)\) is a weight function to be chosen, and \(h_R(t_k)\) is the value of the impulse response function at the chosen instants \(t_k\) \((k = 1, 2, 3, \ldots, m)\). Thus, at least \(m\) discrete values of \(h_R(t)\) are assumed to be known either
from theory or from experiment. The approximation is expected to improve if the order \( n \) of the linear subsystem is increased. A reasonably good approximation is, however, often obtained even if \( n \) is a rather small integer. Remember that \( h_0(t) \), in the integral in eqn (28), is known with limited accuracy in many cases.

Using eqns (29) and (30) in eqn (28), the state-space model of the heaving of the body can be written as

\[
\dot{X}(t) = AX(t) + Bu(t) \quad \text{(35)}
\]

and

\[
y(t) = z(t) = CX(t) \quad \text{(36)}
\]

where the system state vector \( X(t) \), the input function \( u(t) \) and the system matrices \( A, B \) and \( C \) are

\[
X(t) = [X_p(t)^T \quad z(t)^T \quad \hat{z}(t)^T]^T
\]

\[
= [x_1(t) \quad x_2(t) \quad \ldots \quad x_n(t) \quad x_{n+1}(t) \quad x_{n+2}(t)]^T \quad \text{(37)}
\]

\[
u(t) = f_h(t) \quad \text{(38)}
\]

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
A_p & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
C_p/a & b/a & 0 & 0 & \cdots & 0 & 0 & 0 \\
\end{bmatrix} \quad \text{(39)}
\]

\[
B = \begin{bmatrix}
0 & 0 & \cdots & 0 & -1/a \\
\end{bmatrix}^T \quad \text{(40)}
\]

\[
C = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 \\
\end{bmatrix} \quad \text{(41)}
\]

respectively, where \( a = -(M + m_o) \) and \( b = \rho g S \). The initial condition is \( X(0) = [X_p(0)^T \quad z(0) \quad \hat{z}(0)]^T \).

4 STATE-SPACE MODELLING OF THE RADIATION FORCE

As an example, let us consider the heaving motion of a floating cylinder-shaped body with its vertical axis coinciding with the origin \((x=0)\). Let the radius and draught of the cylinder be \( R = 0.35 \text{ m} \) and \( c = 0.63 \text{ m} \), respectively, which means that the body displaces a water volume of \( V = 0.242 \text{ m}^3 \). The water depth is \( h = 3 \text{ m} \). The coefficients of added mass, wave radiation damping and wave excitation force, as obtained by numerical solution of the hydrodynamic boundary value problem, are given in Figs 1 and 2.\(^{33,34}\) The corresponding impulse response function for radiation, as calculated using eqns (25) and (27) is shown in Fig. 3 (solid line). With \( \rho = 1000 \text{ kg/m}^3 \), the mass of the body is \( M = 242 \text{ kg} \) and the frequency independent added mass of the body is \( m_o = 83.5 \text{ kg} \). Non-dimensionalized values of the body’s draught, the water depth and the frequency-independent added mass are \( c/R = 1.80 \), \( h/R = 8.57 \) and \( m_o/M = 0.345 \), respectively. For the radiation problem, a linear sub-system of order two to four is usually good enough to approximate the integral term of eqn (28). In this example, a sub-system of order three is tested. The approximation matrices of the sub-system obtained by minimizing eqn (34) are

\[
A_p = \begin{bmatrix}
0 & 0 & -17.9 \\
1 & 0 & -17.7 \\
0 & 1 & -4.41 \\
\end{bmatrix} \quad \text{(42)}
\]

\[
B_p = \begin{bmatrix}
36.5 & 394 & 75.1 \\
\end{bmatrix}^T \quad \text{(43)}
\]

\[
C_p = \begin{bmatrix}
0 & 0 & 1 \\
\end{bmatrix} \quad \text{(44)}
\]

In the numerical determination of these matrices, a constant weight function \( G(t) = 1 \) has been chosen, and 51 discrete values of \( h(t) \) \((t_k = (k-1)\Delta t, \quad (k = 1, 2, \ldots, 51))\) \((\Delta t = 0.1 \text{ s})\) have been used. Figure 3 also shows the impulse response function corresponding to the state-space model of the sub-system of order three. It can be found from Fig. 3 that the impulse response function...
corresponding to the approximate linear sub-system is in very good agreement with the impulse response function of the radiation force. This means that the state-space model agrees very well with the impulse response function model.

Using eqns (42)–(44) in eqn (39) and considering that the order of the sub-system is three, the system state vector $X(t)$, the input variable $u(t)$ and the system matrices $A$, $B$ and $C$ of the state-space model of the heaving body are

$$X(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) & x_5(t) \end{bmatrix}^T$$

$$A = \begin{bmatrix} 0 & 0 & -17.9 & 0 & 36.5 \\ 1 & 0 & -17.7 & 0 & 394 \\ 0 & 1 & -4.41 & 0 & 75.1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.00307 & -11.6 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.00307 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$$

and

respectively. [SI units are implied with the numbers in eqns (47) and (48).]

Substituting eqns (45)–(48) into eqn (35), and eqns (45) and (49) into eqn (36), the heaving motion of the body can be simulated in the time domain by a state-space model of order five when the excitation force for heave is used as the input.

5 STATE-SPACE MODELLING OF THE WAVE EXCITATION FORCE

As pointed out earlier (see Section 3), the wave excitation force is independent of the motion of the body, and can be considered, instead of the incident wave, as the input to the linear system of the heaving body. Considering the motion of the body as the output, this will be the same whether we use the excitation force or the incident wave as the input. However, for practical systems in ocean engineering, it is more general if the incident waves, rather than the excitation forces, are measured and considered as inputs to the system. To model a dynamic ocean system using the state-space description with the incident wave as the input, an additional linear sub-system, which has the input $u(t) = \eta(0, t) = a(t)$, and an output $y(t) = f_x(t)$, must be built in the form of a state-space description, and the total system in the form of a state-space description can be obtained by combining the sub-system for the excitation force with our previous system using the excitation force as the input [see eqns (35), (36) and (45)–(49)].

Taking the inverse Fourier transform of eqn (18), the wave excitation force in the time domain can be expressed as

$$f_x(t) = \int_{-\infty}^{\infty} h(t - \tau) \eta(0, \tau) \, d\tau = \int_{-\infty}^{\infty} h(t - \tau) a(\tau) \, d\tau$$

(50)

where $\eta(x, t)$ is the incident wave elevation in the time domain, and $a(\tau) = \eta(0, \tau)$ is the wave elevation at the origin. (Note that the incident wave is the undisturbed sea wave, as it would be if the body were absent.) Further, the impulse response function $h_x(t)$ is the inverse Fourier transform of the excitation force coefficient $H_x(\omega)$ for heave [see eqn (18)]. Applying the inverse Fourier transform to the function $H_x(\omega)$ shown in Fig. 1, an impulse response function $h_x(t)$ as shown in Fig. 4 is obtained.

From this, it will appear that $h_x(t)$ is not causal. This can be understood from the fact that the chosen input (the incident wave elevation at the origin, $x = 0$) is not

![Fig. 4](image-url)
the cause of the output (the heave excitation force). The real cause of the output, as well as the input, may be a distant storm (or a wave-maker in the laboratory). The generated wave may hit the body and exert a wave force before any wave has reached the conveniently chosen reference point \( x = 0 \). As \( h(t) \) is non-causal, the linear sub-system of the excitation force in the form of a state-space description is expressed as [see eqns (14) and (15)]

\[
\dot{X}_r(t) = A_f X_r(t) + B_f u(t) \quad \text{with} \quad X_r(0) = 0
\]

and

\[
y_f(t) = C_f X_r(t + t_c) \approx \int_{-\infty}^{\infty} h_f(t - \tau)\eta(0, \tau) \, d\tau
\]

where \( X_r(t) \), \( u(t) = \eta(0, t) \) and \( y_f(t) \approx \eta_f(t) \) are the state vector, the input variable and the output variable of the linear sub-system, respectively. Moreover, \( t_c \) is the causalizing time shift of \( h(t) \) [see eqns (4), (6) and (15)]. The system matrices \( A_f \), \( B_f \) and \( C_f \) have the same forms as in eqns (31), (32) and (33), and they can be determined by minimizing the following target function [see eqn (34)]

\[
Q = \sum_{k=1}^{m} G(k) |h_f(t_k) - C_f e^{A_f t_c} B_f|^2
\]

where \( h_f(t) = h_f(t - t_c) \) is the causalized impulse response function corresponding to \( h(t) \).

It can be observed from Fig. 4 (solid line) that \( h(t) \) is negligible for \( t < -1.2 \text{ s} \). So, the causalizing time shift \( t_c \) is set as \( t_c = 1.2 \text{ s} \), corresponding to a non-dimensionalized value \( t_{ca} = t_c g/R^{0.5} = 6.35 \). Figure 4 also shows the causalized impulse response function \( h_f(t) \) of the excitation force (dashed line).

A linear sub-system of order five is tested to approximate the wave excitation force. The matrices of the sub-system obtained by minimizing eqn (53) are (in SI units)

\[
A_f = \begin{bmatrix}
0 & 0 & 0 & 0 & -409 \\
1 & 0 & 0 & 0 & -459 \\
0 & 1 & 0 & 0 & -226 \\
0 & 0 & 1 & 0 & -64.0 \\
0 & 0 & 0 & 1 & -9.96
\end{bmatrix}
\]

(54)

\[
B_f = \begin{bmatrix}
1549866 \\
-116380 \\
24748 \\
-644 \\
19.3
\end{bmatrix}^T
\]

(55)

and

\[
C_f = \begin{bmatrix}
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(56)

In the computation for minimizing eqn (53), a constant weight function \( G(t_k) = 1 \) was chosen, and 45 discrete values of \( h_f(t_k) = h_f[(k-1)\Delta t, \Delta t] \), \( (k = 1, 2, \ldots, 45) \), \( \Delta t = 0.1 \text{ s} \) were used. It can also be seen from Fig. 4 that the causalized impulse response function \( \mathcal{O}(\xi) \) of the excitation force corresponding to the state-space model of order five agrees very well with the result (dashed line) corresponding to the inverse Fourier transform of the frequency response function of the excitation force. This means that a state-space model can also be applied to approximate a non-causal linear system.

6 STATE-SPACE MODELLING OF THE HEAVE-VS-WAVE RESPONSE

For a given amplitude \( A(\omega) \) of incident wave, a body’s heave response is given by eqn (22) together with eqn (18). The corresponding transfer function \( Z(\omega)/A(\omega) \) is shown by the curves in Fig. 5, which have also been obtained by using the numerical information represented in Figs 1 and 2. However, a response function as represented by Fig. 5 could alternatively have been obtained by experiment.

By taking the inverse Fourier transform of the body’s heave transfer function, the corresponding computed impulse response function \( h(t) \) is obtained as shown by the curve in Fig. 6. It appears that \( h(t) \) is non-causal, as was to be expected, because of the non-causality of the impulse response function \( h_f(t) \) for the excitation force. For \( t < -t_c \), where \( t_c = 1.2 \text{ s} \), we should expect \( h(t) = 0 \). The small oscillations seen in the graph in Fig. 6, for \( t < -t_c \) are due to numerical inaccuracies such as, for example, truncation at high frequencies. We note that the graph in Fig. 6 shows the typical behaviour of a transient damped free oscillation, for which the time between consecutive maxima is approximately 1.8 s. This corresponds to a natural angular frequency \( \omega_n \approx 3.4 \text{ rad/s} \). This parameter matches the resonance condition

\[
\omega_n^2[M + m(\omega_0)] - \rho g S - \rho g \pi R^2
\]

(57)

where numerical values of \( M \), \( \rho \) and \( R \) are given in Section 4, and where values for \( m(\omega) \) may be obtained from Fig. 2. Note that the condition of eqn (57) corresponds to setting \( \xi_c = 0 \) and \( N = 0 \) in eqn (21).

The total system model of the heaving body in the
Fig. 6. Non-dimensional impulse response function $h_z(t)$ of the heave motion of the body considered as the output when the incident wave elevation at $x=0$ is the input. The curve is obtained by taking the numerical inverse Fourier transform of the transfer function shown by the curve in Fig. 5. The dots show results obtained from an approximate fifth-order state-space model.

Form of a state-space description by using the incident wave as the system’s input can be obtained by combining eqns (51)-(52) with eqns (35), (36) and (45)-(49). The total system now has an order of ten. Note that the sub-system of the excitation force has an order of five and it can be solved independently. So, the total system is, in fact, expressed by two systems of order five, in which, one (the excitation force) can be solved independently, and the other (the heave motion including the effects of radiation) can be solved by using the output of the first system as its input.

A method similar to that of Section 5 may, however, be applied to build a state-space model directly from the impulse response function. In this way, it may well be possible to approximate the heave response by means of a state-space model of order less than ten. Explicitly, we apply eqns (51)-(53) with the subscript $f$ replaced by subscript $z$. For our computation, we choose a state-space order of five, a weighting function $G(t)$, and the 80 discrete values $h_z(\tau_k) = h_z(\tau_k - t_c)$, where $\tau_k = (k-1)\Delta t$, $k=1, 2, 3, \ldots, 80$; $\Delta t=0.1$ s; and $t_c=1.2$ s. We obtain the following system matrices

$$A_z = \begin{bmatrix} 0 & 0 & 0 & 0 & -192.34 \\ 0 & 0 & 0 & 1 & -178.04 \\ 0 & 0 & 1 & 0 & -77.14 \\ 0 & 0 & 0 & 1 & -27.24 \\ 0 & 0 & 0 & 0 & -5.12 \end{bmatrix}$$  

$$B_z = \begin{bmatrix} 194.49 \\ -60.46 \\ 9.72 \\ -1.29 \\ -0.012 \end{bmatrix}^T$$  

and

$$C_z = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Application of this approximate fifth-order state-space model gives results as shown (by the dots) in Figs 5 and 6. We observe that reasonably good results are obtained with this approximate model. It is remarkable that the fit in Fig. 6 is also good for the later parts of the impulse response function, in spite of the fact that only the early part (the first 8 s, or 42.3 units of normalized time) was used for determining the $2\pi=10$ unknown coefficients by minimizing the target function $Q$ according to eqn (53), with subscript $f$ replaced by $z$. Admittedly, the computed values of $h_z(t)$ show that the approximation is better in the early part of the diagram shown in Fig. 6 than in the later part, although the graph’s resolution does not permit the reader to see this in the diagram.

The previously described tenth-order state-space model, consisting of two decoupled fifth-order state-space models for the excitation force vs the incident wave (Section 5) and the heave motion vs the excitation force (Section 4), has also been investigated, and it gives a better approximation than our fifth-order state-space model [eqns (58)-(60)]. Contrary to Fig. 5, the deviation between the dots and curves for $\omega(R/g)^{1/2} < 1$ would not have been visible in the diagram if the tenth-order model had been applied instead of the fifth-order model. However, in most cases, the approximation represented by the dots in Fig. 5 is acceptable. Thus in such cases, the fifth-order state-space model may be adopted. Note that this model was based on the curve given in Fig. 6 without using the state-space models obtained in Sections 4 and 5.

7 CONCLUSION

Various methods to analyze linear systems have been reviewed. Particular attention has been paid to systems which are not causal. The description of systems in terms of transfer functions in the frequency domain and impulse response functions in the time domain has been used for quite some time in ocean engineering. A purpose of this paper has been to promote the application of state-space modelling in hydrodynamics. Such modelling will be convenient for time-domain simulation and digital control and optimization problems in ocean engineering. In many cases, for instance, in relation to ship motion or wave energy conversion, the hydrodynamic interaction is only part of the total system, and a state-space model will be very useful to combine the hydrodynamic part with other parts, possibly non-linear, of the total system.

While this paper has considered linear systems with a single input and a single output, it is possible to generalize to systems with several inputs and outputs.

The method applied in this paper to construct a state-space model for a system seems to be more straightforward in use than the techniques discussed recently by Jefferys and Goheen.15 This comparison refers to modelling of the radiation problem.
We have also shown how to model non-causal systems where the excitation force or the body's motion is the output when the wave elevation of an incident wave is the input. This non-causality means that if real-time control is to be applied, it is necessary to have some future knowledge of the incident wave at the body. However, in this paper, the problem of wave prediction has not been addressed, since the used input functions are either a sinusoidal function or an impulse function of time.

State-space modelling is applicable not only for wave-energy converters, but also for ships and other dynamic systems in ocean engineering. For controlling a large ship, it may be acceptable to approximate the hydrodynamic parameters to constants, corresponding to their low-frequency limits (see, for instance, Fossen). However, for vessels which are small compared to the wavelength, it is much more important to take into account the variation of the hydrodynamic parameters. For such cases, the methods presented in this paper may be of great value.

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